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The Double Tangents of a Binodal Quartic.

By H. Bateman.

1. The number of conditions required to determine an algebraic curve C_n of the nth degree is usually said to be $\frac{1}{2}n(n+3)$. This statement, however, is not complete, for a number of questions must be settled before we can be sure that the theorem is true in a particular case. It will be sufficient at present to consider the following questions:

1) Are the conditions consistent with the properties of a C_n?

2) Are the conditions independent of one another?

3) Is there a non-degenerate C_n satisfying the given conditions?

Of course it would be useless to try to determine a nodal cubic with three inflexions not on a line,* and in this case the reason is quite clear, but there are other cases in which it is not easy to see that the conditions are inconsistent or not independent.

For instance nine double points should determine a C_6, but Cayley† and Halphen‡ have shown that if the points are chosen arbitrarily, the C_6 consists of a double cubic through the nine points; on the other hand if the points are not chosen arbitrarily and there is one non-degenerate C_6 having the given double points, there is an infinite number of C_6's forming a pencil, which satisfy the conditions.

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*A cubic satisfying the conditions would necessarily consist of three straight lines. It should be noticed that to be given an inflexion on a cubic counts as two conditions, but to be given three inflexions on a line is equivalent to only five. For five conditions are sufficient to make two of the points inflexions on a cubic and to make the third point lie on the cubic; this third point is then necessarily an inflexion.

It can easily be verified in fact that there are ∞^1 cubics with a given double point and three given inflexions on a line. For the cubic

$$xy(y-mx)+ax^2+2hxy+by^2=0$$

has inflexions at infinity in the directions $x=0$, $y=0$, $y-mx=0$ if the *two* conditions

$$2h+\frac{a}{m}=0, \qquad 2h+bm=0$$

are satisfied. The equation thus contains the arbitrary parameter h.

†*Proc. London Math. Soc.*, Vol. III (1869–71), pp. 198–202. A corresponding theorem for quartic surfaces had been given previously, *ibid.*, p. 19.

‡*Bull. de la Société Mathématique de France* (1882).

Other theorems of a similar type have been given for curves of higher degree by Halphen and de Jonquières.* In some cases a C_n can be found to satisfy more than the proper number of conditions and then we must conclude that the conditions are not independent.

For instance if we are given six tangents to a conic, two quartic curves can be found with these lines as double tangents and a given point O as double point.† Again, if $A_1, A_2, \ldots, A_6$ are six points on a conic, we can describe a quartic curve to have a double point at a given point O and to touch OA_1, $OA_2, \ldots, OA_6$ at the points $A_1, A_2, \ldots, A_6$.‡

In each case the quartic can apparently be made to satisfy fifteen conditions instead of fourteen. Now this occurs very frequently in the case of the quartic curve, and I have found that when double points and bitangents are given, the method of counting constants must be used with caution; for instance, we have the following theorems:

If we are given four double tangents and one double point of a binodal quartic, the other double point can be chosen arbitrarily when the double tangents are such that a conic passes through the points of contact of three of them§ *and not through the points of contact of the fourth.*

If the points of contact of the double tangents lie on a conic, the second double point can not be chosen arbitrarily; it must lie on one of three conics. When the double point has been chosen in a suitable way, there are ∞^1 *quartic curves with the given double points and double tangents.*

If the double tangents are such that no set of three are syzygetic,‖ *the second double point can not be chosen arbitrarily, and when the double points have been suitably chosen, there are* ∞^1 *quartics which satisfy the conditions.*

The properties of the quartic on which these theorems depend are not altogether new, but I have not seen a statement of them in the above form. I have discussed the theorems from several points of view and have shown that the second theorem may be derived from Cayley's theorem by a (2, 1) transformation. The second theorem is also discussed with the aid of a particular (2, 1) transformation. The properties of this transformation seem so interesting that some of them have been developed for their own sake.

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* *Comptes Rendus,* Vol. CV, pp. 971–978.

† E. Duporcq: *Nouvelles Annales,* Ser. 4, t. 2, p. 161.

‡ E. Bertini: *Acc. R. de Lincei* (3), I (1877), pp. 92–97.

§ When this is the case the three double tangents are said to form a syzygetic set. Frobenius, *Crelle's Journal,* Bd. XCIX. See also a paper by Miss Long, *Proc. London Math. Soc.,* Ser. 2, Vol. IX (1911), p. 205.

‖ They are said to form an asyzygetic set.

Two of the three theorems can be proved geometrically with the aid of the theorem that the eight bitangents of a bicircular quartic touch a rational curve of the third class. This proof did not occur to me until the paper was almost completed.

2. The properties of the double tangents of a binodal quartic may be studied by taking the case of the bicircular quartic and considering this as the stereographic projection of a cyclic from a point O on the sphere S containing the cyclic.*

Let H_1, H_2, H_3, H_4 be the four quadric cones through the cyclic, V_1, V_2, V_3, V_4 their vertices and F_1, F_2, F_3, F_4 the projections of these vertices on the plane π, which is, of course, parallel to the tangent plane at O to the sphere S. We shall call the line of intersection of a plane through O with π the *trace* of this plane.

The traces of the tangent planes from O to the four cones H_1, H_2, H_3, H_4 are double tangents of the bicircular quartic, and so the double tangents meet in pairs in the points F_1, F_2, F_3, F_4 which are the centers of the four circles which invert the bicircular quartic into itself.†

Let h_1, h_2, h_3, h_4 be the four conics which are the reciprocals of the cones H_1, H_2, H_3, H_4 with regard to the sphere S, and let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be their projections on the plane π. If h_1 meet T, the tangent plane at O to S in the points L_1, M_1, the points L_1, M_1 will be the poles with regard to S of the tangent planes from O to the cone H_1. It follows then that the lines OL_1, OM_1 are perpendicular to the traces of these planes on the plane π, but OL_1, OM_1 are parallel to the asymptotes of the conic Γ_1; hence the two double tangents through F_1 are perpendicular to the asymptotes of the conic Γ_1. The conics $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are known to be confocal. This may be proved as follows:

Since the cones H and the sphere S belong to a linear system of two terms the tangential equations of the conics h_r are of the form

$$\Sigma + \lambda_r U = 0 \qquad (r = 1, 2, 3, 4),$$

where Σ is the tangential equation of the sphere S.

The poles of a plane through O with regard to these conics lie on a line which meets T in the pole of the plane with regard to S. It follows then that if B is the trace of the plane through O and b the projection of the line, B and b

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* See for instance Darboux's book, "Une classe remarquable de courbes et de surfaces algébriques."

† This theorem and some of the following ones are due to E. Czuber, *Zeitschr. für Math. und Physik*, Vol. XXXII (1887), p. 267. I have amplified his method of treatment and applied it to the theorems given by Richter.

are perpendicular and b is the locus of the poles of B with regard to the conics $\mathbf{\Gamma}$. Now when a system of conics are such that the poles of a line B lie on a perpendicular line b for all positions of B, the conics are necessarily confocal; therefore $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$, $\mathbf{\Gamma}_3$, $\mathbf{\Gamma}_4$ are confocal conics.*

If F_1X, F_1X' are the two double tangents through F_1, F_2X, F_2X' the two double tangents through F_2, their four points of intersection lie on a circle C_{12}. For we may regard the two pairs of lines as two conics whose axes are parallel to the axes of the conics $\mathbf{\Gamma}$, and when the axes of two conics are parallel their four points of intersection lie on a circle.

Richter † has shown that the six circles of type C_{12} are concentric with the conics $\mathbf{\Gamma}$. To prove this by stereographic projection we start with the equations of the four cones H_r in the form

$$S + \lambda_r Q = 0 \qquad (r = 1, 2, 3, 4).$$

The equations of the pairs of tangent planes from O to the cones are then given by

$$\lambda_r q (S + \lambda_r Q) = (T + \lambda_r P)^2,$$

where q is the constant obtained by substituting the coordinates of O in the equation of Q, and $T = 0$, $P = 0$ are the polar planes of O with regard to $S = 0$, $Q = 0$ respectively.

The lines of intersection of the two pairs of planes

$$\lambda_1 q (S + \lambda_1 Q) = (T + \lambda_1 P)^2,$$
$$\lambda_2 q (S + \lambda_2 Q) = (T + \lambda_2 P)^2$$

lie on the cone

$$\lambda_1 \lambda_2 (qS - 2PT) = (\lambda_1 + \lambda_2) T^2,$$

whose trace on the plane π ($T = \text{const.}$) is the circle C_{12}. The center of this circle does not depend upon the quantities λ_1, λ_2, and so the six circles of type C_{12} are concentric. If we had drawn tangent cones from O to any two quadrics through the cyclic, their traces on the plane π would be conics touching the bicircular quartic in four points, and the foregoing analysis indicates that the four points of intersection of these conics ‡ lie on a circle concentric with the circles C_{12}. To prove that the circles C_{12} are concentric with the conics $\mathbf{\Gamma}$, we consider the point t which is the pole of T with regard to the conic h_1; this point will project into the center of $\boldsymbol{\Gamma}_1$.

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* The relation between the conics Γ and the points F is such that $F_2F_3F_4$ is a self-polar triangle for Γ_1, $F_3F_1F_4$ a self-polar triangle for Γ_2 and so on. The foci of the conics Γ are pairs of isogonal conjugate points with regard to the pedal triangle of $F_1F_2F_3$.

† *Zeitschr. für Mathematik und Phys.*, Vol. XXXV, Supplement (1890), p. 1.

‡ The axes of the conics are parallel.

Since h_1 is the reciprocal of H_1 with regard to S, it follows that the polar plane of t with regard to S is the same as the polar plane of O with regard to H_1. Consequently

$$S'p = s'P,$$

where S' is the polar plane of t with regard to S, and s', p are the values of S' and P at O. Now the equation of the polar plane of t with regard to the cone

$$\lambda_1\lambda_2(qS - 2PT) = (\lambda_1 + \lambda_2)T^2$$

is

$$\lambda_1\lambda_2(qS' - pT - tP) = (\lambda_1 + \lambda_2)tT,$$

and since $p \equiv q$, $t \equiv s'$ this reduces simply to $T = 0$. It follows, then, that the trace of this cone is a circle whose center is the projection of the point t; hence the circles C and the conics $\boldsymbol{\Gamma}$ are concentric. Richter has also shown that if U_1, U_1' are the feet of the perpendiculars from the center of the circles C on the two double tangents through F_1, then the four lines U_1U_1', U_2U_2', U_3U_3', U_4U_4' meet in a point.

3. When the bicircular quartic is projected into a binodal quartic, the theorems of Richter and Czuber take the following form:

The double tangents of a binodal quartic can be arranged in four pairs, P_1, P_1'; P_2, P_2'; P_3, P_3'; P_4, P_4', meeting respectively in four points F_1, F_2, F_3, F_4. The points of intersection of any two pairs such as P_1, P_1'; P_2, P_2' lie on a conic C_{12} which passes through the double points $\Omega\Omega'$.

The points F_1, F_2, F_3, F_4 are the poles of $\Omega\Omega'$ with regard to the four conics which pass through $\Omega\Omega'$ and the intersections of the tangents from Ω and Ω' to the quartic.

The six conics of type C_{12} have double contact at the points $\Omega\Omega'$.

It follows from the first theorem that if we are given the four double tangents $P_1P_1'P_2P_2'$ and one double point Ω, then the other double point Ω' can not be chosen arbitrarily, for it must lie on a conic through Ω and four of the intersections of the lines $P_1P_1'P_2P_2'$. There are really three conics which correspond to different sets of two out of three pairs of opposite corners of the quadrilateral formed by the double tangents.

We shall now show that this theorem may be deduced from Halphen's theorem on the double points of a sextic. Let seven arbitrary points be chosen as the base points of a cubic $(1,2)$ transformation between two planes X' and X.

This transformation* is such that a point in X' corresponds to two points in X, while a point in X corresponds to only one point in X'. A line in X' corresponds to a cubic curve through the seven base points A_1, A_2, ..., A_7, while a line in X corresponds to a rational cubic touching the synoptic curve J' at six points.

It has been shown by Clebsch, Noether and de Paolis that the synoptic curve J' is a general quartic,† and the corresponding curve J (the Jacobian) is a sextic with double points at $A_1, \ldots, A_7$. Now let us apply this transformation to Halphen's theorem that if eight double points of a sextic are given, the ninth can not be chosen arbitrarily, its locus being in fact a C_9 with triple points at the eight given points.

Let $A_1, A_2, \ldots, A_7, A_8$ be the eight given double points; then if we assume that the sextic is not described by pairs of complementary points in the plane X, the corresponding curve in the plane X' will be a binodal quartic touching the synoptic curve J' at eight points which correspond to those in which the sextic meets J.

The corresponding theorem is that for a certain set of binodal quartics touching J' at eight points the two nodes can not be chosen arbitrarily, and that when the nodes are properly chosen, there are ∞^1 quartics of the set having the given nodes.‡

If A'_8 is the point corresponding to an eighth given double point on Halphen's sextics, the tangents from A'_8 to J' will correspond to the nodal cubics through the eight points, and the points of contact of these tangents to the twelve critical centers for cubics through the eight points. The curve of the ninth degree, which is the locus of a ninth double point on the sextic, meets J at the twelve critical centers and corresponds to a curve of degree

$$3 \times 9 - 7 \times 3 = 6$$

touching J' at the twelve points of contact of the tangents from A'_8. Hence, if one node of the binodal quartic be given, the locus of the other is a sextic curve.

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* The properties of this transformation are discussed very fully by de Paolis. *Memorie dell' Accademia dei Lincei*, Ser. 3, Bd. I–II (3 papers). Miss Scott's papers, *Quarterly Journal*, Vol. XXIX, p. 329, Vol. XXXII, p. 209, *Bull. Amer. Math. Soc.*, Vol. VII, p. 24 (1901), deal with the general theory ot the rational transformation between two planes and will provide the reader with practically all the information that is required here. We may also refer to R. Sturm, *Die Lehre von den geometrischen Verwandtschaften*, Vol. IV, pp. 232–268.

† This may be seen at once by mapping the plane X on to a cubic surface by means of cubics through A_1, A_2, ..., A_6. The synoptic curve in X' can then be regarded as the section of the tangent cone to the cubic from the point corresponding to A_7.

‡ I notice that on p. 278, Vol. IV, of his "Geometrische Verwandschaften," Sturm obtains the theorem on the cubic surface which corresponds to Halphen's theorem; as the above theorem may be deduced from Sturm's theorem by projection, it is probable that the theorem was known to Sturm.

To investigate the question analytically, let the equation of the binodal quartic be

$$Q(x', y', z') = 0.$$

When x', y', z' are expressed in terms of (x, y, z), this equation must take the form

$$Q \equiv SS',$$

where S is the equation of the sextic and S' the equation of the complementary sextic. Now S and S' each have double points at the seven base points and meet on J, hence S, S' and J belong to the same pencil and we may put

$$S \equiv H + \lambda J,$$
$$S' \equiv H - \mu J,$$

where λ and μ are constants.

Since the synoptic curve J' corresponds to J, we have

$$J'(x', y', z') \equiv [J(x, y, z)]^2,$$

and x, y, z can be expressed rationally in terms of x', y', z' and $\sqrt{J'}$. When these rational values are substituted in the expressions for S and S' and J is put equal to $+\sqrt{J'}$ in one case and equal to $-\sqrt{J'}$ in the other, the two resulting expressions must be the same, for both S and S' correspond to the same curve Q. This, however, is only possible if $\lambda = \mu$ and H is a rational function of (x', y', z'). The equation for Q now takes the form

$$Q = H^2 - \lambda^2 J^2 = H^2 - \lambda^2 J' = 0,$$

where $H(x', y', z')$ is of the second degree in (x', y', z'). The eight points of contact of the binodal quartic with J' thus lie on a conic $H = 0$.

Let us now change the notation and use (x, y, z) to denote homogeneous coordinates in the plane X', also let $F(x, y, z) = 0$ be the equation of J', so that $J' \equiv F$. If (x, y, z), (x', y', z') are two double points of the curve Q, we have:

$$\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial z} = 0,$$

$$\frac{\partial Q'}{\partial x'} = \frac{\partial Q'}{\partial y'} = \frac{\partial Q'}{\partial z'} = 0.$$

These equations give

$$2H\frac{\partial H}{\partial x} = \lambda^2\frac{\partial F}{\partial x}, \text{ etc.};$$

therefore

$$2H\left(x'\frac{\partial H}{\partial x} + y'\frac{\partial H}{\partial y} + z'\frac{\partial H}{\partial z}\right) = \lambda^2\left(x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial z}\right),$$

and similarly

$$2H'\left(x\frac{\partial H'}{\partial x'} + y\frac{\partial H'}{\partial y'} + z\frac{\partial H'}{\partial z'}\right) = \lambda^2\left(x\frac{\partial F'}{\partial x'} + y\frac{\partial F'}{\partial y'} + z\frac{\partial F'}{\partial z'}\right).$$

Now since H is of the second degree,

$$x'\frac{\partial H}{\partial x} + y'\frac{\partial H}{\partial y} + z'\frac{\partial H}{\partial z} = x\frac{\partial H'}{\partial x'} + y\frac{\partial H'}{\partial y'} + z\frac{\partial H'}{\partial z'};$$

hence

$$H\left(x\frac{\partial F'}{\partial x'} + y\frac{\partial F'}{\partial y'} + z\frac{\partial F'}{\partial z'}\right) = H'\left(x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial z}\right),$$

and consequently

$$F\left(x\frac{\partial F'}{\partial x'} + y\frac{\partial F'}{\partial y'} + z\frac{\partial F'}{\partial z'}\right)^2 = F'\left(x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial z}\right)^2.$$

This relation shows that the two double points can not be chosen arbitrarily; when one double point A_8' is given, the locus of the other is a sextic curve which touches J' at the points where it is met by the first polar of J', *i. e.*, at the points of contact of the tangents from A_8'. The sextic has a triple point at A_8' and double points at the three points of intersection of the first polar and polar line of P' with regard to J'.

The case in which J' consists of four straight lines is of special interest, for then the lines form a set of double tangents of the binodal quartic whose points of contact lie on a conic, *i. e.*, a syzygetic set according to the nomenclature of Frobenius.

Putting $F = xyzt$, the equation of the sextic becomes

$$x'y'z't'[t(x'yz + y'zx + z'xy) + xyzt']^2 = xyzt[t'(xy'z' + yz'x' + zx'y') + x'y'z't]^2,$$

or

$$(t'x'yz - txy'z')(t'y'zx - tyz'x')(t'z'xy - tzx'y') = 0.$$

The sextic thus consists of three conics, each of which passes through the two double points and two pairs of opposite corners of the quadrilateral formed by the double tangents. This corresponds exactly with Richter's theorem.

If then we are given four double tangents which form a syzygetic set and one double point of a binodal quartic, the other double point can not be chosen arbitrarily; its locus consists of three conics. When the two double points have been suitably chosen, there are ∞^1 binodal quartics with the given double points and double tangents.*

If we are given the two double points and three double tangents whose points of contact lie on a conic, the fourth double tangent of the syzygetic set must pass through one of three fixed points on the line joining the nodes.

When Hesse's notation for the double tangents is used, the eight double tangents of a binodal quartic can be represented by the symbols †

$$\begin{array}{cccc} 12 & 27 & 28 & 35 \\ 78 & 18 & 17 & 46 \end{array}$$

A pair such as 12, 78 meet in a point F_1; any two pairs $\genfrac{}{}{0pt}{}{27}{18}$ and $\genfrac{}{}{0pt}{}{28}{17}$ form a syzygetic set. There are clearly six syzygetic sets. The tangents from the nodes each count as two double tangents and have two symbols associated with them,‡ thus

$$\binom{48}{68} \quad \binom{14}{26} \quad \binom{24}{26} \quad \binom{67}{47}$$

are the tangents from one node, and

$$\binom{51}{31} \quad \binom{52}{32} \quad \binom{57}{37} \quad \binom{58}{38}$$

are the tangents from the other. The line joining the nodes is represented by the four symbols

$$34, \quad 36, \quad 45, \quad 46.$$

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*In the case of the bicircular quartic the double tangents F_1X, F_1X', F_2X, F_2X' determine the circle C_{12} and its center K. The asymptotes of the conics Γ_1 and Γ_2 can then be determined, but the linear dimensions of one of these conics can be chosen arbitrarily. When Γ_1 is known, the line F_3F_4, which is the polar of F_2, can be found; Γ_2 is then determined by the fact that F_1 is the pole of F_3F_4 with regard to Γ_2. Finally F_3, F_4 are determined by the fact that they are conjugate with regard to Γ_1 and Γ_2.

† Equations of the 28 double tangents of a quartic and the corresponding symbols are given by Cayley (*Crelle's Journal* (1883), p. 93; *Collected Papers*, Vol. XI, p. 221, Vol. XII, p. 74), and by de Paolis, *Rend. Lincei*, II, 2 (1877–1878).

‡ This is well known. It may be deduced, for instance, from the results given by G. Fonténé. Sur les dégénérescences des 63 systemes de coniques quadruplement tangentes a une quartique. *Bull. de la Société Mathématique de France*, t. 27 (1899), p. 229.

4. We must now investigate the question whether the double points can be chosen arbitrarily when we are given four double tangents which do not form a syzygetic set. There are two cases to be considered.

(1) We may be given four double tangents which are such that the points of contact of no set of three lie on a conic.

(2) We may be given four double tangents which are such that a conic passes through the points of contact of three of them, but does not pass through the points of contact of the fourth.

In the first case the double tangents will belong to one of the sixteen sets:

12	18	17	35	78	27	28	35
12	18	17	46	78	27	28	46
12	27	28	35	78	18	28	35
12	27	28	46	78	18	28	46
12	27	18	35	78	18	17	35
12	27	18	46	78	18	17	46
12	27	17	35	78	27	17	35
12	27	17	46	78	27	17	46

formed by taking one double tangent from each pair. This case may be discussed by means of the particular (2, 1) cubic transformation in which the synoptic curve consists of four straight lines.

Let the net of cubics in the plane X pass through the fixed point O and touch three fixed lines OA, OB, OC at the fixed points A, B, C respectively. Taking ABC as triangle of reference and choosing the coordinates so that O is the point $(1, 1, 1)$, the net of cubics is represented by the equation

$$\xi x^2(y-z)+\eta y^2(z-x)+\zeta z^2(x-y)=0.$$

This corresponds to the line

$$\xi x'+\eta y'+\zeta z'=0$$

if

$$\frac{x'}{x^2(y-z)}=\frac{y'}{y^2(z-x)}=\frac{z'}{z^2(x-y)}.$$

These are the equations of the transformation; the Jacobian consists of the six lines

$$xyz(y-z)(z-x)(x-y)=0,$$

and the synoptic curve of the four lines

$$x'=0,\ \ y'=0,\ \ z'=0,\ \ x'+y'+z'=0.$$

A point (x', y', z') in the plane X' corresponds to two points (x, y, z), (x_1, y_1, z_1), where

$$x_1 : y_1 : z_1 = \frac{x}{zx + xy - yz} : \frac{y}{xy + yz - zx} : \frac{z}{yz + zx - xy}.$$

Let us now consider the curve which corresponds to a quartic having a double point at O and touching OA, OB, OC at A, B, C respectively. The equation of such a quartic is

$$\begin{aligned} &lx^2(y-z)(2x-y-z) + my^2(z-x)(2y-z-x) \\ &\quad + nz^2(x-y)(2z-x-y) - \lambda x^2(y-z)^2 - \mu y^2(z-x)^2 - \nu z^2(x-y)^2 = 0, \end{aligned}$$

and the equation of the corresponding curve is found as follows: Put

$$\frac{x'}{x^2(y-z)} = \frac{y'}{y^2(z-x)} = \frac{z'}{z^2(x-y)},$$

$$\theta = mz' - ny' - \lambda x', \quad \phi = nx' - lz' - \mu y', \quad \psi = ly' - mx' - \nu z';$$

then the previous equation may be written in the form:

$$\frac{\theta x'}{x^2} + \frac{\phi y'}{y^2} + \frac{\psi z'}{z^2} = 0.$$

Now

$$\frac{x'}{x^2} + \frac{y'}{y^2} + \frac{z'}{z^2} = 0,$$

$$\frac{x'}{x} + \frac{y'}{y} + \frac{z'}{z} = 0;$$

therefore on eliminating x, y, z we obtain the equation

$$\sqrt{x'(\phi - \psi)} + \sqrt{y'(\psi - \theta)} + \sqrt{z'(\theta - \phi)} = 0. \tag{1}$$

This represents a quartic curve having the point $\theta = \phi = \psi$ as a node and the lines $x' = 0$, $y' = 0$, $z' = 0$, $x' + y' + z' = 0$ as double tangents. In order to make the point (x', y', z') a node, we have then to satisfy only two equations, viz.:

$$\theta = \phi = \psi.$$

Now the original quartic curve will have a node at the point (x, y, z) if three conditions

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

are satisfied. These, together with the two conditions $\theta = \phi = \psi$, give five linear conditions for the determination of the ratios of $(l, m, n, \lambda, \mu, \nu)$. When these are determined, the quartic curve (1) will have a double point at (x', y', z') and a second double point corresponding to (x, y, z), both points being arbitrary. To show that the quartic is not degenerate, we have to prove that a proper quartic can be drawn to have a double point at O, to touch OA, OB, OC at A, B, C, to have a second double point at $P(x, y, z)$, and to pass through two assigned complementary points corresponding to (x', y', z'). It is certainly possible to determine a quartic from these conditions, for we have only

$$3 + 6 + 3 + 2 = 14$$

conditions to satisfy; but there is a cubic curve touching OA, OB, OC at A, B, C respectively and passing through O, P and the two complementary points, and it is this cubic curve together with the line OP which is determined by the given conditions. If P is the double point of a proper quartic curve which satisfies the conditions, there will be a pencil of quartics of the same kind.

It follows, then, that for a non-degenerate curve of type

$$\sqrt{x'(\phi - \psi)} + \sqrt{y'(\psi - \theta)} + \sqrt{z'(\theta - \phi)} = 0$$

the two double points can not be chosen arbitrarily, and when the double points have been suitably chosen there are ∞^1 curves with the given double points and double tangents whose equations have the above form.

To find if the points of contact of the double tangents lie on a conic or not, we put $x' = 0$, $y' = 0$, $z' = 0$, $x' + y' + z' = 0$ in succession. The points of contact of these four double tangents are then found to be given by the equations

$$\begin{aligned}
&y'^2(l + n) - z'^2(m + l) - y'z'(\nu + m - n + \mu) = 0, \qquad x' = 0;\\
&z'^2(l + m) - x'^2(m + n) - z'x'(\lambda + \nu + n - l) = 0, \qquad y' = 0;\\
&x'^2(m + n) - y'^2(l + n) - x'y'(\mu + \lambda + l - m) = 0, \qquad z' = 0;\\
&y'^2(\lambda + \mu) + z'^2(\lambda + \nu) + 2\lambda y'z' = 0, \qquad x' + y' + z' = 0.
\end{aligned}$$

It is clear from the form of the coefficients of x'^2, y'^2, z'^2 in the first three equations that a conic does not pass through the points of contact of the first three double tangents, and it is easy to verify that the points of contact of any other set of three do not lie on a conic. For instance, the general equation of a conic through the points of contact of the second and third is

$$\begin{aligned}
y'^2(l + n) + z'^2(l + m) - x'^2(m + n) + x'y'(\mu + \lambda + l - m)&\\
-z'x'(\lambda + \nu + n - l) + ky'z' = 0;&
\end{aligned}$$

this meets the last double tangent $x' + y' + z' = 0$ in points given by

$$z'^2(\lambda + \nu) - y'^2(\lambda + \mu) + (k + \nu - \mu - m - n - 2l)y'z' = 0,$$

and under no circumstances are these the same as the points of contact which are given by the equation

$$y'^2(\lambda+\mu)+z'^2(\lambda+\nu)+2\lambda y'z'=0.$$

Hence we have the following theorem:

If we are given an asyzygetic set of double tangents of a binodal quartic, the two double points can not be chosen arbitrarily, and when the double points have been suitably chosen there are ∞^1 *quartics with the given double points and double tangents.*

Let (X', Y', Z') be the coordinates of the double point given by the equations $\theta=\phi=\psi$. Then we have

$$\begin{aligned} X'(m+n)-Y'(\mu+l)+Z'(\nu-l)&=0,\\ X'(\lambda-m)+Y'(n+l)-Z'(\nu+m)&=0,\\ -X'(\lambda+n)+Y'(\mu-n)+Z'(l+m)&=0. \end{aligned}$$

Also, if (x', y', z') are the coordinates of a second double point, we find on differentiating the equation

$$\sqrt{x'(\phi-\psi)}+\sqrt{y'(\psi-\theta)}+\sqrt{z'(\theta-\phi)}=0$$

with regard to x', y', z', that

$$\begin{aligned} &\frac{1}{2\sqrt{x'}}\sqrt{(\phi-\psi)}+(m+n)\sqrt{\frac{x'}{\phi-\psi}}+(\lambda-m)\sqrt{\frac{y'}{\psi-\theta}}-(\lambda+n)\sqrt{\frac{z'}{\theta-\phi}}=0,\\ &\frac{1}{2\sqrt{y'}}\sqrt{(\psi-\theta)}-(\mu+l)\sqrt{\frac{x'}{\phi-\psi}}+(n+l)\sqrt{\frac{y'}{\psi-\theta}}+(\mu-n)\sqrt{\frac{z'}{\theta-\phi}}=0,\\ &\frac{1}{2\sqrt{z'}}\sqrt{(\theta-\phi)}+(\nu-l)\sqrt{\frac{x'}{\phi-\psi}}-(\nu+m)\sqrt{\frac{y'}{\psi-\theta}}+(l+m)\sqrt{\frac{z'}{\theta-\phi}}=0. \end{aligned}$$

Multiplying these equations by X', Y', Z' respectively and adding, we get

$$\frac{X'}{\sqrt{x'}}\sqrt{(\phi-\psi)}+\frac{Y'}{\sqrt{y'}}\sqrt{(\psi-\theta)}+\frac{Z'}{\sqrt{z'}}\sqrt{(\theta-\phi)}=0.$$

Combining this with the equation

$$\sqrt{x'(\phi-\psi)}+\sqrt{y'(\psi-\theta)}+\sqrt{z'(\theta-\phi)}=0,$$

we get

$$\frac{\sqrt{(\phi-\psi)}}{(Y'z'-Z'y')\sqrt{x'}}=\frac{\sqrt{(\psi-\theta)}}{(Z'x'-X'z')\sqrt{y'}}=\frac{\sqrt{(\theta-\phi)}}{(X'y'-Y'x')\sqrt{z'}}.$$

Therefore

$$x'(Y'z'-Z'y')^2+y'(Z'x'-X'z')^2+z'(X'y'-Y'x')^2=0. \qquad (2)$$

If (X', Y', Z') be given, the locus of (x', y', z') is a rational cubic having a double point at (X', Y', Z') and passing through the six corners of the quadrilateral formed by the four lines

$$x'=0, \quad y'=0, \quad z'=0, \quad x'+y'+z'=0.$$

If, on the other hand, the point (x', y', z') be given, the locus of (X', Y', Z') is a conic which consists of a pair of straight lines through the point (x', y', z'). It is in fact part of the Jacobian of the net of cubics passing through the six corners of the quadrilateral and the point (x', y', z').

The two double points of the binodal quartic thus bear different relations to the four double tangents of an asyzygetic set; this is suggested by the nomenclature, for an asyzygetic set

$$12, \quad 18, \quad 17, \quad 35$$

appears to be associated with the node [35] and the set

$$12, \quad 18, \quad 17, \quad 46$$

with the node [46].

The lack of symmetry is also indicated by the corresponding configuration in the X plane, for one double point corresponds to a pair of complementary points M, M' and the other to a node P on the original quartic.

To find the condition that a quartic may touch OA, OB, OC at A, B, C, pass through M, M' and have a node at P, we must find the condition corresponding to

$$x'(Y'z'-Z'y')^2+y'(Z'x'-X'y')^2+z'(X'y'-Y'x')^2=0.$$

Putting

$$\frac{x'}{x^2(y-z)}=\frac{y'}{y^2(z-x)}=\frac{z'}{z^2(x-y)},$$

$$\frac{X'}{X^2(Y-Z)}=\frac{Y'}{Y^2(Z-X)}=\frac{Z'}{Z^2(X-Y)},$$

the equation takes the form

$$(y-z)(z-x)(x-y)[X'yz+Y'zx+Z'xy][X'yz(yz-zx-xy) + Y'zx(zx-xy-yz)+Z'xy(xy-yz-zx)]=0.$$

Hence, when M and M' are given, the locus of P is a conic * and a quartic.

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* If the conic pass through a fixed point (a, b, c), the cubic (2) will also pass through a fixed point (a', b', c') and its node lie on the fixed line

$$\frac{X'}{a}+\frac{Y'}{b}+\frac{Z'}{c}=0.$$

There is a (1, 2) correspondence between the point (a', b', c') and this line, in which corresponding elements are incident.

When P is given the locus of M and M' consists of two cubics, each of which is described by pairs of complementary points. If we are given two double points and three asyzygetic double tangents, a fourth double tangent forming with the given ones an asyzygetic set can not be chosen arbitrarily. To find the envelope of the fourth double tangent, we write lx, my, nz in place of x', y', z'; we then have to find the envelope of the double tangent

$$lx + my + nz = 0,$$

when l, m, n are subject to the relation

$$mnx(Yz - Zy)^2 + nly(Zx - Xy)^2 + lmz(Xy - Yx)^2,$$

which is deduced from (2) by putting

$$lX = X', \ mY = Y', \ nZ = Z'.$$

The envelope of the fourth double tangent is thus a conic which touches the sides of the triangle $xyz = 0$. This conic passes through the point (x, y, z) and also through the point $\left(\frac{X^2}{x}, \frac{Y^2}{y}, \frac{Z^2}{z}\right)$. The tangents at these two points meet at the point (X, Y, Z). The conic, therefore, touches the line joining the nodes at one of the nodes.* In reality there are two conics, one being associated with each node.

5. Turning now to the second case when we are given three double tangents forming a syzygetic set, we shall next consider whether a fourth double tangent can be chosen arbitrarily when the two double points are given, and whether the double points can be chosen arbitrarily when a fourth double tangent is given. Let the three given bitangents be taken as sides of the triangle of reference; then there are three possible forms for the equation of a binodal quartic, viz.:

$$[yz + x(lx + my + nz) + (px + qy + rz)^2]^2 = 4\,xyz(lx + my + nz),$$
$$[zx + y(lx + my + nz) + (px + qy + rz)^2]^2 = 4\,xyz(lx + my + nz),$$
$$[xy + z\,(lx + my + nz) + (px + qy + rz)^2]^2 = 4\,xyz(lx + my + nz).$$

In the first case the double points are given by the equations

$$px + qy + rz = 0,$$
$$yz - x(lx + my + nz) = 0.$$

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*This might have been deduced from the fact that a Brianchon set of six bitangents of a general quartic touch a conic. When four asyzygetic bitangents and one node A are given, the locus of the other node consists either of the tangents to the two conics through A which touch the four lines, or else of the rational cubic which is the locus of the points of contact of the tangents from A to conics touching the four lines.

Let the coordinates of the double points be (a, b, c), (α, β, γ) respectively; then

$$a\beta\gamma(la + mb + nc) = abc(l\alpha + m\beta + n\gamma).$$

Hence the double tangent $lx + my + nz = 0$ passes through a fixed point on the line of nodes. This agrees with the result of § 3. Writing the equation of the quartic in the form

$$[yz - x(lx + my + nz)]^2 + (px + qy + rz)^2[(px + qy + rz)^2 + 2yz + 2x(lx + my + nz)] = 0, \quad (1)$$

we may transform it into

$$\begin{aligned}[][yz - x(lx + my + nz) + \theta(px + qy + rz)^2]^2 \\ = (px + qy + rz)^2[(\theta^2 - 1)(px + qy + rz)^2 \\ + 2(\theta - 1)yz - 2(\theta + 1)x(lx + my + nz)].\end{aligned}$$

Now let θ be chosen so that

$$(\theta^2 - 1)(px + qy + rz)^2 + 2(\theta - 1)yz - 2(\theta + 1)x(lx + my + nz) = 0$$

breaks up into factors; then the lines represented by these equations will be double tangents. Let $x = \xi y + \eta z$ be one of these lines. Then we must have identically

$$\begin{aligned}(\theta^2 - 1)[(p\xi + q)y + (p\eta + r)z]^2 + 2(\theta - 1)yz \\ - 2(\theta + 1)(\xi y + \eta z)[(l\xi + m)y + (l\eta + n)z] \equiv 0;\end{aligned}$$

$$\therefore \quad (\theta^2 - 1)(p\xi + q)^2 - 2(\theta + 1)(\xi l + m)\xi = 0,$$

$$2(\theta^2 - 1)(p\xi + q)(p\eta + r) + 2(\theta - 1) - 2(\theta + 1)[\xi(l\eta + n) + \eta(l\xi + m)] = 0,$$

$$(\theta^2 - 1)(p\eta + r)^2 - 2(\theta + 1)(l\eta + n)\eta = 0.$$

These equations give

$$\theta = 1 + 2\frac{\xi(\xi l + m)}{(p\xi + q)^2} = 1 + 2\frac{\eta(\eta l + n)}{(p\eta + r)^2}.$$

Substituting these values of $\xi l + m$, $\eta l + n$ in the second equation, we get

$$2(\theta + 1)(p\xi + q)(p\eta + r) + 2 - (\theta + 1)\left[\frac{\xi}{\eta}(p\eta + r)^2 + \frac{\eta}{\xi}(p\xi + q)^2\right] = 0,$$

or

$$\frac{2\xi\eta}{\theta + 1} = [\xi(p\eta + r) - \eta(p\xi + q)]^2 = (\xi r - \eta q)^2.$$

Hence we have the equations

$$\frac{\xi\eta}{(\xi r - \eta q)^2} = 1 + \frac{\xi(\xi l + m)}{(p\xi + q)^2} = 1 + \frac{\eta(\eta l + n)}{(p\eta + r)^2}. \quad (2)$$

Now, if the double points are given, the equations

$$pa + qb + rc = 0,$$
$$p\alpha + q\beta + r\gamma = 0$$

give

$$\frac{p}{b\gamma - c\beta} = \frac{q}{c\alpha - a\gamma} = \frac{r}{a\beta - b\alpha} = \lambda,$$

where λ is an arbitrary parameter, and the equations

$$bc - a(la + mb + nc) = 0,$$
$$\beta\gamma - \alpha(l\alpha + m\beta + n\gamma) = 0$$

give

$$m(b\gamma - c\beta) = \frac{c\gamma}{a\alpha}(b\alpha - a\beta) + l(\alpha c - a\gamma),$$

$$n(b\gamma - c\beta) = \frac{b\beta}{a\alpha}(\gamma a - c\alpha) - l(b\alpha - a\beta).$$

Hence p, q, r, l, m, n can be expressed in terms of two arbitrary parameters l and λ. These parameters both occur in equations (2) and these equations can be solved for l and λ; consequently there are two arbitrary parameters in the determination of ξ and η, and so the double tangent $x = \xi y + \eta z$ can be chosen arbitrarily. It follows, then, that when this double tangent is given, the two double points can be chosen arbitrarily.

To complete the proof we must show that the binodal quartic represented by equation (1) is not degenerate. If it consisted of a straight line and a cubic, the line would have to be the line joining the double points, viz., $px + qy + rz = 0$. This is not generally a factor of (1), so this possibility may be rejected. If the curve consisted of two conics, there would only be four constants in the equation instead of six, so this possibility may also be rejected.

6. The theorems of §§ 3, 4 and 5 may also be proved geometrically by considering the case of a bicircular quartic.

It is known that in the case of a general quartic the twelve bitangents of a Steiner system, *e. g.*,

$$\begin{array}{cccccc} 12 & 27 & 28 & 35 & 36 & 34 \\ 78 & 18 & 17 & 46 & 56 & 54 \end{array}$$

touch a curve of the third class,* and that the six points of intersection of the pairs of bitangents lie on a conic.† Now, when the quartic has double points at the circular points, the four bitangents 36, 34, 56, 54 coincide with the line at infinity, and so the curve of the third class has this line as a double tangent. The points of intersection of 34, 54 and 36, 56 become the points of contact of the double tangent; these points lie on a conic through F_1, F_2, F_3, F_4, and this conic is a rectangular hyperbola since F_1, F_2, F_3, F_4 form a triangle and its orthocenter. Hence, the points of contact of the line at infinity with the curve are harmonic with the circular points. They are also the points at infinity on the bisectors of the angles between pairs of bitangents. To prove this let $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ be three pairs of bitangents; then the curve of the third class is the Cayleyan of the net of conics

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*Salmon: Higher plane curves, 3rd edition (1879), pp. 150, 226.

†Baker: *Proc. London Math. Soc.*, Ser. 2, Vol. IX (1911), p. 145.

$$\lambda\alpha\alpha' + \mu\beta\beta' + \nu\gamma\gamma' = 0.$$

But since the circle through the four points of intersection of $\alpha\alpha'$ with $\beta\beta'$ is concentric with the circle through the four points of intersection of $\alpha\alpha'$ with $\gamma\gamma'$, the problem of finding the Cayleyan may be simplified by regarding the net of conics as built up from

$$ax^2 + by^2 = 0, \quad (x-\alpha_0)^2 + (y-\beta_0)^2 = r^2, \quad (x-\alpha_0)^2 + (y-\beta_0)^2 = \rho^2.$$

The tangential equation of the Cayleyan is easily found to be

$$(b\xi^2 + a\eta^2)(\beta_0\xi - \alpha_0\eta) = (a-b)\xi\eta\zeta.$$

Its form indicates at once that the line at infinity ($\xi = \eta = 0$) is a double tangent and that the points of contact are on the axes, *i. e.*, on the bisectors of the angles between the bitangents $ax^2 + by^2 = 0$.

By using the equation of the Cayleyan we can express the equations of the eight bitangents in the form

$$\left.\begin{aligned} y + xt_s &= \frac{(a+bt_s^2)(\beta_0 t_s - \alpha_0)}{(a-b)t_s}, \\ y + xt_s &= \frac{(a+bt_s^2)(\beta_0 t_s + \alpha_0)}{(a-b)t_s}, \end{aligned}\right\} \qquad (s = 1, 2, 3, 4)$$

where

$$t_1^2 t_2^2 t_3^2 t_4^2 = -\frac{a^2\alpha_0^2}{b^2\beta_0^2}.$$

The coordinates of the points of intersection of pairs of bitangents are

$$x_s = \frac{\alpha_0}{b-a}\left(b + \frac{a}{t_s^2}\right), \quad y_s = \frac{\beta_0}{a-b}(a + bt_s^2), \quad (s = 1, 2, 3, 4)$$

and these form a triangle and its orthocenter on account of the relation between t_1, t_2, t_3, t_4. The equation of the circle through the four points of intersection of the bitangents with parameters $t_r, \ldots, t_r$ with those with parameters $t_s, \ldots, t_s$ is

$$(x-\alpha_0)^2+(y-\beta_0)^2=(a^2\alpha_0^2+b^2\beta_0^2t_r^2t_s^2)(1+t_r^2)(1+t_s^2).$$

Now let us see how the Cayleyan helps us in the problem of determining the bicircular quartic by means of four bitangents. If four syzygetic bitangents such as

$$\begin{matrix} 12 & 27 \\ 78 & 18 \end{matrix}$$

are given, the bisectors of the angles between each pair must be parallel, otherwise there would be no quartic. These four bitangents determine the Cayleyan completely; a third bitangent, such as 28, may be taken to be an arbitrary tangent of the Cayleyan, and then the configuration can be determined completely, for 17 is the tangent making the same angles with the axes as 28, and three points such as F_1, F_2, F_3 determine the fourth on account of the property of the orthocenter.

If a set of bitangents such as

$$\begin{matrix} 12 & 27 & 28 \\ 78 & & \end{matrix}$$

be given, the Cayleyan can be determined, for 12 and 78 determine the origin of coordinates, the directions of the axes and the ratio a/b, while 27 and 28 give the constants α and β. The whole configuration is now determined uniquely; hence four double tangents of this kind can be chosen arbitrarily, and there is only one bicircular quartic.

The case when four bitangents such as

$$12 \quad 27 \quad 28 \quad 35$$

are given can not be discussed very easily with the aid of the Cayleyan.

7. The special (2, 1) transformation used in this paper is capable of so many useful applications[*] that it seems worth while to work out its properties in detail. The equations

$$\frac{x'}{x^2(y-z)}=\frac{y'}{y^2(z-x)}=\frac{z'}{z^2(x-y)}$$

give

$$x'(y+z)+y'(z+x)+z'(x+y)=0.$$

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[*] I have used it in a former paper, *Quarterly Journal*, No. 147 (1906), to prove an interesting property of a uninodal quartic.

This is the equation of the line joining two complementary points. On the portion $x=0$ of the Jacobian, we have

$$x'=0, \quad \frac{y'}{y}=-\frac{z'}{z};$$

hence the line joining two consecutive corresponding points is in this case

$$x(Y-Z)-yZ+zY=0.$$

This passes through the point $(-1, 1, 1)$. The "principal lines" at points of $y=0$, $z=0$ can be shown to pass through $(1, -1, 1)$, $(1, 1, -1)$ respectively.

A straight line $lx+my+nz=0$ corresponds to a rational cubic, for the equations

$$lx+my+nz=0,$$
$$x(y'+z')+y(z'+x')+z(x'+y')=0$$

give

$$\frac{x}{n(z'+x')-m(x'+y')}=\frac{y}{l(x'+y')-n(y'+z')}=\frac{z}{m(y'+z')-l(z'+x')}.$$

Substituting in the identical relation

$$\frac{x'}{x}+\frac{y'}{y}+\frac{z'}{z}=0,$$

we obtain

$$\frac{x'}{n(z'+x')-m(x'+y')}+\frac{y'}{l(x'+y')-n(y'+z')}+\frac{z'}{m(y'+z')-l(z'+x')}=0.$$

This equation represents a cubic having a node at the point given by

$$\frac{y'+z'}{l}=\frac{z'+x'}{m}=\frac{x'+y'}{n}.$$

The curve touches the three lines $x'=0$, $y'=0$, $z'=0$ and meets them again at the points where they are met by the line $x'+y'+z'=0$. The cubic has a cusp if the line $lx+my+nz=0$ passes through one of the points $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$.

If the line passes through the point $(1, 1, 1)$, the cubic breaks up into the line $x'+y'+z'=0$ and a conic touching the four lines

$$x'=y'=z'=x'+y'+z'=0.$$

For if the equation of the straight line be written in the form

$$l(y-z)+m(z-x)+n(x-y)=0,$$

we have

$$\frac{lx'}{x^2}+\frac{my'}{y^2}+\frac{nz'}{z^2}=0.$$

But

$$\frac{x'}{x^2}+\frac{y'}{y^2}+\frac{z'}{z^2}=0,$$

$$\frac{\frac{x'}{x^2}}{m-n}=\frac{\frac{y'}{y^2}}{n-l}=\frac{\frac{z'}{z^2}}{l-m}.$$

Substituting in the equation

$$\frac{x'}{x}+\frac{y'}{y}+\frac{z'}{z}=0,$$

we get

$$\sqrt{x'(m-n)}+\sqrt{y'(n-l)}+\sqrt{z'(l-m)}=0.$$

The complete curve in the plane X corresponding to this conic consists of a straight line and a quintic curve whose equation is

$$\begin{aligned}(m-n)x[xy+yz-zx]\,[yz+zx-xy]&\\ +(n-l)y[zx+xy-yz]\,[yz+zx-xy]&\\ +(l-m)z[xy+yz-zx]\,[zx+xy-yz]&=0.\end{aligned}$$

This curve passes through the point O (1, 1, 1) and has tacnodes at A, B, C, the tangents at these tacnodes being OA, OB, OC. We shall now show that a straight line through the point O is cut in a constant anharmonic ratio by this quintic curve. Consider two conics Q and Q' touching the four lines

$$x'=y'=z'=x'+y'+z'=0.$$

By a known theorem,* the cross-ratio of the four points on Q in which it is cut by Q' is equal to the cross-ratio of the four points on Q' at which it is touched by the common tangents of Q and Q'.

Now let L and H be the line and quintic corresponding to Q, L' and H' the line and quintic corresponding to Q'. The four points of intersection other than O of L and H' correspond to the four points of intersection of Q and Q', and so the cross-ratio of these four points on L is equal to the cross-ratio of the four points on Q, for when there is a (1, 1) correspondence between the points of a line and conic, cross-ratios are unaltered by the transformation.

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*This is proved by reciprocating one conic into the other.

Again, the cross-ratio of the four points in which Q' touches the lines

$$x' = y' = z' = x' + y' + z' = 0$$

is equal to the cross ratio $(Oabc)$, where a, b, c are the points in which L' meets the lines BC, CA, AB. Hence, it follows that the cross-ratio in which L is cut by H' is equal to $(Oabc)$ and is independent of the position of L.

To verify this theorem analytically, we write

$$Y \equiv x - y, \quad Z \equiv x - z.$$

The equation of the quintic then becomes

$$\begin{aligned}(m-n)x[x^2 - 2xY + YZ]\,[x^2 - 2xZ + YZ]& \\ + (n-l)(x-Y)(x^2 - YZ)(x^2 - 2xZ + YZ)& \\ + (l-m)(x^2 - 2xY + YZ)(x^2 - YZ) &= 0,\end{aligned}$$

or

$$\begin{aligned}x^4(bY + cZ) - 4x^3(b+c)YZ + 2x^2[YZ^2(2b+c) + Y^2Z(b+2c)]& \\ - 4x(b+c)Y^2Z^2 + Y^2Z^2(bY + cZ) &= 0,\end{aligned}$$

where

$$b = n - l, \quad c = l - m.$$

Calculating the invariants of this quartic in x, we have,

$$\begin{aligned}3I &= 4Y^2Z^2(Y-Z)^2(b^2 + bc + c^2), \\ 27J &= 4Y^3Z^3(Y-Z)^3(2b+c)(b+2c)(b-c).\end{aligned}$$

Hence, $\dfrac{I^3}{27J^2}$ is independent of the ratio of Y to Z. Since the cross-ratios of the four values of x depend only on the ratio $\dfrac{I^3}{J^2}$, it follows that the anharmonic ratio is the same for all straight lines through O.